

CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences
Pasadena, California 91125

PRICE UNCERTAINTY AND THE EXHAUSTIVE FIRM*

H. Stuart Burness
Economics Department
University of Kentucky
and
Environmental Quality Laboratory
California Institute of Technology

Financial support from the Institute of Mining and Minerals Research, University of Kentucky, and from the Energy Research Development Association, Grant No. EX-76-C-03-1305, California Institute of Technology Energy Research Program is gratefully acknowledged. I wish to thank Stuart Greenbaum, Søren Lemche and Agnar Sandmo for helpful comments, and Tracy Lewis for a suggestion that greatly improved the exposition of this paper. Any remaining inadequacies are my own.

Social Science Working Paper

Number 156

April 1977

PRICE UNCERTAINTY AND THE EXHAUSTIVE FIRM

In an earlier edition of this journal [2] the risk-neutral exhaustive¹ firm's reactions to various tax-subsidy schemes were explored. Sandmo [10] analyzed the output effects of uncertainty on risk-averse competitive firms and derived comparative statics results dependent on attitudes towards risk. In this paper, we examine the risk-averse exhaustive firm and find: (1) some results are independent of risk aversion assumptions; (2) comparative statics results differ from the no resource constraint case; and (3) results depend on the relative magnitude of the discount rate and fixed costs. It is assumed that capital markets are imperfect, or, alternatively, that if perfect capital markets exist, firms do not have access to them; otherwise risk-aversion would not be operative relative to production decisions as firms would simply extract the resource so as to maximize the discounted present value of profits and then go to the capital market to obtain their desired income stream. This assumption seems to closely approximate real world situations for some resource industries; in particular the coal industry is characterized by a predominance of equity funding, particularly for smaller firms, indicating inaccessibility to capital markets. We find here that even in a simple model, lack of access to perfect capital markets leads to ambiguities and qualitative differences vis-a-vis the risk-neutral firm.

¹The modifier exhaustive is employed to designate the price-taker facing a resource constraint while competitive is used in its received sense (i.e., a price-taker facing no resource constraints).

The analysis is developed step-wise with increasing complexity in order to illustrate the fundamental concepts. We focus on the role of concave utility; concave utility drives the results, and, while price uncertainty is the source of risk aversion here, it may arise from other forms of uncertainty or even other manifestations of price uncertainty. The deterministic case is presented first for later comparison. Concave utility is then introduced in the absence of price uncertainty. Finally the case of price uncertainty and risk aversion is considered. A summary integrates and interprets the results in the text. Proofs of theorems are all relegated to the Appendix.

I. The Deterministic Case

The theory of exhaustible resources has been elaborated by Hotelling [6], Scott [12], Gordon [3], Smith [13], Cummings and Burt [3], Kuller and Cummings [7], Schulze [11] and by a number of authors in a recent symposium [15]. Some of these results are reproduced for comparison with cases to be examined later.

Letting $x(t)$ be rate of extraction, $C[x(t)]$, the costs of producing $x(t)$; $p(t)$, the price of output² and K the total stock of reserves, the problem is to maximize aggregate profits, $V(\pi)$, where

$$(I.1) \quad V(\pi) = \int_0^T e^{-rt} [px - C(x)] dt$$

²We assume that market price is independent of remaining reserves and not directly dependent on time; while this does not preclude dependence on market output we find that the ambiguity of results in the analysis of the firm suggests a market analysis would not be too rewarding.

subject to

$$(I.2) \quad \dot{y} = -x, y(0) = K, y(T) = 0$$

$y(t)$ being the reserves remaining at time t , and T being the date of depletion. This is a standard variational problem, and necessary conditions require that along the optimal path, x_d

$$(I.3) \quad e^{-rt}[p - C'(x)] = \lambda$$

where λ is the (constant) multiplier associated with (I.2), and transversality conditions require that, for $t = T$,

$$(I.4) \quad \frac{C(x)}{x} = C'(x)$$

where $C'(x) = \frac{\partial C}{\partial x}$. It is implicitly assumed that costs consist of both variable and fixed components and fixed costs may be avoided through exit from the industry.³ As usual, it is supposed that C is monotonically increasing and convex. For the case of a zero discount rate (I.3) and (I.4) require that optimal production occurs at the point of minimum average cost. Rather than maximizing profit per time period, producers maximize profit per ton of output (average profit) and in this manner attain a global maximum for the profits attributable to their stock of reserves.

³Fixed costs of this sort were identified by Smith [14]. When a positive rate of discount is employed, fixed costs are not problematical (see Burness [2]).

The reaction of the exhaustive firm to various tax-subsidy policies differ from the traditional competitive results.⁴ Propositions 1.1-1.3 present the deterministic results for the exhaustive firm.⁵

Proposition I.1: Suppose the producers of a non-renewable resource is a value maximizer; the imposition of a franchise law (constant over time) results in hastened depletion.

Proposition I.2: Suppose the producer of a non-renewable resource is a value maximizer: (i) if fixed costs are positive and the discount rate is zero, the imposition of a severance tax (constant over time and of either ad valorem or unit type) causes no change in extraction plans; (ii) if fixed costs are non-negative and the discount rate is positive, the imposition of a severance tax results in postponed depletion.

Proposition I.3: Suppose the producer of a non-renewable resource is a value maximizer: (i) if fixed costs are positive and the discount rate is zero, the imposition of a profits tax (constant over time) causes no change in extraction plans; (ii) if fixed costs are non-negative and the discount rate is positive, the imposition of a profits tax results in postponed depletion.

Figures I-1 through I-3 depict these results. Proofs and interpretations of these propositions plus their generalizations to the case where the rate of taxation is variable can be found in Burness [2].

⁴We suppose that the exhaustive firm accepts prices as parametric thus endowing it with one dimension analagous to the competitive paradigm. We do not specify the entry-exit properties as they are not germane to our analysis. However, in view of the irreversible nature of capital outlays for exhaustive firms, the high degree of capital intensity and the presence of uncertainty, one suspects that entry and exit may not be free. Hendry [5] has some observations on this matter.

⁵Since subsidies can be viewed as negative taxes, results are presented for tax policies exclusively.

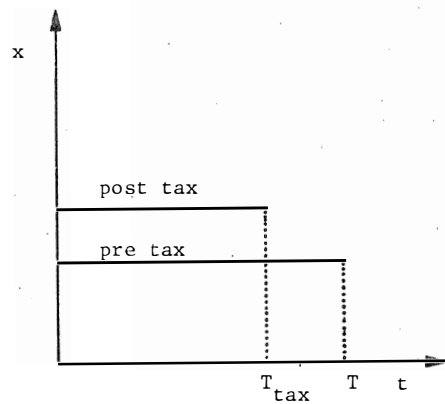


Figure I.1a: The effect of a franchise tax on a value maximizer $F > 0$ and $r = 0$.

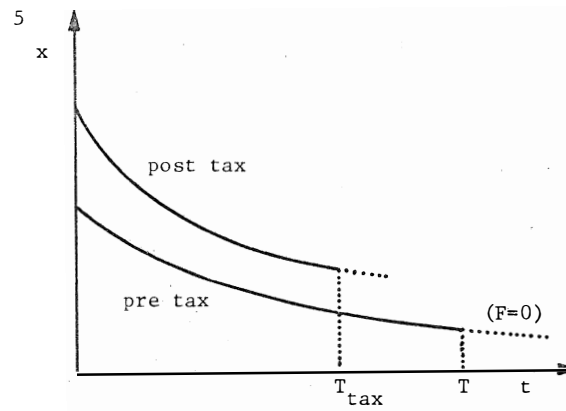


Figure I.1b: The effect of a franchise tax on a value maximizer: $F \geq 0$ and $r > 0$.

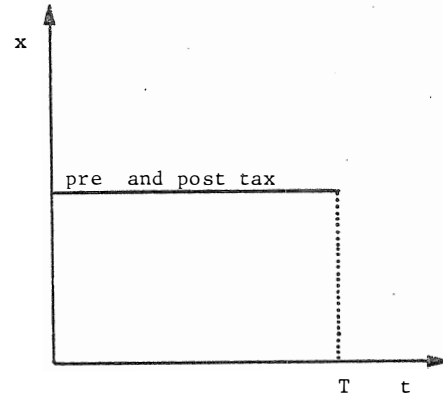


Figure I.2a: The effect of a severance tax on a value maximizer: $F > 0$ and $r = 0$.

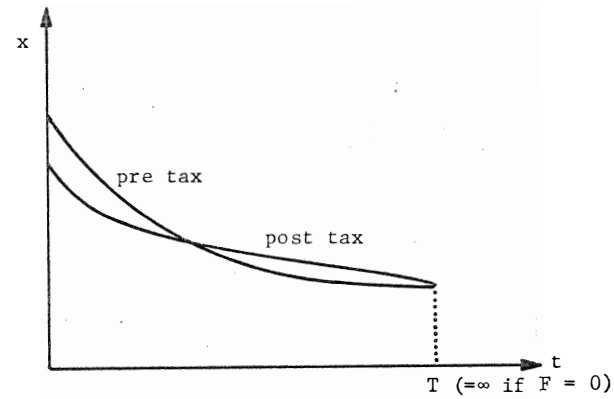


Figure I.2b: The effect of a severance tax on a value maximizer: $F \geq 0$ and $r > 0$.

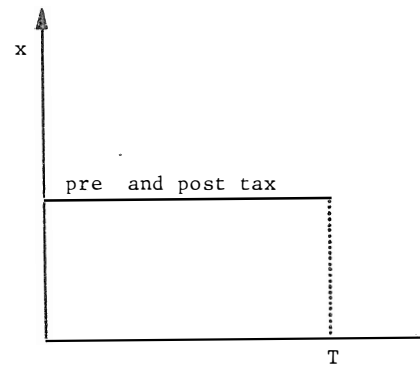


Figure I.3a: The effect of a profit tax on a value maximizer: $F > 0$ and $r = 0$.

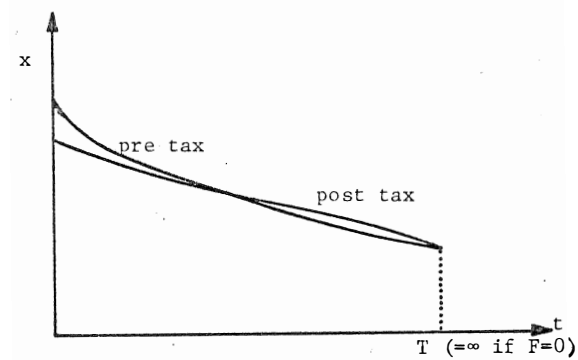


Figure I.3b: The effect of a profit tax on a value maximizer: $F \geq 0$ and $r > 0$.

II. Utility Maximization

For purposes of comparison we first recall the utility maximizing paradigm for a competitive (no resource constraint) producer.

A. No Resource Restraint

While Sandmo's analysis was essentially static, it is easily dynamized, and in fact, the static formulation is representative of the dynamic problem since, under the assumption of stationary period-utility, behavior is the same in all time periods. If the aggregate utility of profits is given by

$$(II.1) \quad W = \int_0^{\infty} e^{-rt} u[\pi(t)] dt$$

then utility maximizing competitive output must satisfy

$$(II.2) \quad u'(\pi) [p - C'(x)] = 0$$

where a prime denotes differentiation.⁶ Since the marginal utility of income is positive, (II.2) requires that price equals marginal cost. Consequently competitive utility maximizing output equals competitive profit maximizing output in all time periods.

B. An Exhaustible Resource

The presence of a resource constraint fundamentally alters the formulation. To avoid some ethical problems associated with resource depletion we assume that the resource under consideration is inessential: it is not necessary for survival. Also the firm's inability to gain

⁶Throughout, it is assumed that $u(\pi)$ is twice continuously differentiable, possesses a third derivative, and that $u(\pi)$ is monotonic increasing and concave.

access to perfect capital markets implies that the producer maximizes the utility of profits. If such access were available the producer would merely maximize the present value of earnings and borrow or lend as necessary to achieve his desired consumption pattern.

Two distinct intervals of production can be imagined. On the first, positive production and non-negative profits are observed while on the second, after the resource is depleted, zero profits and the associated level of utility are experienced. Thus the producer chooses an extraction path, x , and depletion date, T , so as to maximize.

$$(II.3) \quad W^* = \int_0^T e^{-rt} u(\pi) dt + \int_T^{\infty} e^{-rt} u_0 dt$$

subject to (I.2), where T is the date of resource depletion,⁷ and

$$u_0 = u(\pi) |_{\pi=0}.$$

Necessary conditions require that for an interior solution

$$(II.4) \quad e^{-rt} u'(\pi) [p - C'(x)] = \lambda$$

along the utility maximizing arc, \hat{x} , while transversality conditions require that

$$(II.5) \quad e^{-rt} \left\{ u'(\pi) [p - C'(x)] - \frac{u(\pi) - u_0}{x} \right\} = 0$$

for $t = T$. For the case of a zero discount rate equation (II.5) states that the marginal utility of profits obtained from incremental changes

⁷Equation (II.3) implicitly suggests that the firm is infinitely lived. While no definitive answer can be given one must admit the possibility of finite lifetime. The argument to be presented below is still valid provided T^* , the date the firm ceases to exist, exceeds T , the date of optimal resource depletion. However if $T^* < T$ then the problem becomes a fixed horizon problem and many of the results differ. The anomaly arises from the assumed absence of capital markets. We restrict the analysis to the case where $T < T^* < \infty$.

in production must equal average utility (per unit of output); (II.5) also characterizes a maximum of average utility (per unit of output) and this maximum is unique, with depletion occurring in a finite time interval (for $r = 0$ and $F > 0$). Note that the tenets of von Newman-Morgenstern measurable utility are satisfied, for if $v(\pi) = au(\pi) + b$, then along the utility maximizing arc

$$(II.6) \quad e^{-rT} v'(\pi) [p - C'(x)] = e^{-rT} \frac{v(\pi) - v_0}{x}$$

and (II.6) reduces immediately to (II.5).

III. Concave Utility

The assumption of risk aversion is tantamount to the presence of concave utility and uncertainty. In this section we employ a convenient separation and consider concave utility when price is known. This approach is followed primarily for pedagogical purposes as it indicates further lack of symmetry with the competitive paradigm; in particular, utility maximizing exhaustive output generally differs from profit maximizing exhaustive output, while for the competitive case, utility and profit maximizing output are identical (as shown in II.A). To unify the analysis we maintain the neoclassical assumption on utility functions; i.e., $\lim_{z \rightarrow 0} u'(z) = \infty$ and $\lim_{z \rightarrow \infty} u'(z) = 0$.

A. Utility Maximizing Output

Since u is unique up to linear monotonic transformations, the value of u_0 is in essence arbitrary. Consequently we set $u_0 = 0$ so as to avoid carrying a constant through the analysis. In this case (II.5)

becomes

$$(III.1) \quad u'(\pi) [p - C'(x)] = \frac{u(\pi)}{x}$$

for $t = T$.

Theorem III.1: Suppose the producer of a non-renewable resource is a utility maximizer ($u'' < 0$). Then the utility maximizing producer postpones depletion relative to the value maximizing producer.⁸ Specifically (i) if fixed costs (F) are zero and the discount rate (r) is positive and $x_d(t)$ and $\hat{x}(t)$ are the optimal value and utility maximizing extraction schedules, there exists a t^* quite that $x_d(t) > \hat{x}(t)$ for $t < t^*$ and $x_d(t) < \hat{x}(t)$ for $t > t^*$; (ii) if $F > 0$ and $r = 0$, then $x_d(t) = \bar{x}_d$, $\hat{x}(t) = \bar{x}$ and $\bar{x}_d > \bar{x}$; (iii) if $F > 0$, $r > 0$ and T_d and \hat{T} are the optimal value and utility maximizing depletion dates, then $T_d < T$ and $x_d(T_d) > \hat{x}(\hat{T})$. In addition if \hat{x} and x_d intersect then x_d intersects \hat{x} from above.

Figure III.1(i) through III.1(iii) depict these results. Note that the ultimate relationship of the two paths depends on the relative magnitude of fixed costs and the discount rate.

Corollary III.1: Let $R_A(\pi) = -u''(\pi)/u(\pi)$, the measure of absolute risk aversion. If fixed costs are zero, a more risk averse producer, in the sense that $R_A^1(\pi) > R_A^2(\pi)$, tends to postpone extraction relative to the plans of a less risk averse producer; i.e., there exists a t^* such that $\hat{x}^2(t) > \hat{x}^1(t)$ for $t < t^*$ and $\hat{x}^2(t) < \hat{x}^1(t)$ for $t > t^*$.

This result is depicted in Figure III.1(iv).

⁸A result similar to this has been shown by Lewis [8] in the context of an infinite horizon discrete time planning model.

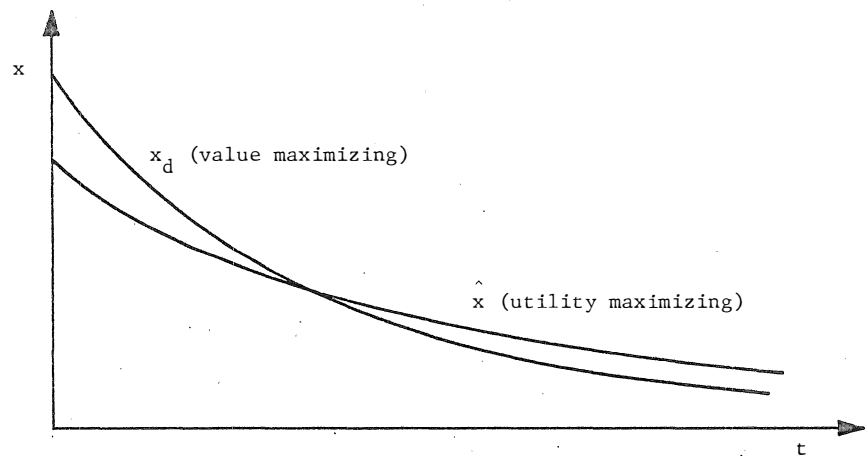


Figure III.1(i): Value and utility maximizing paths when $F = 0$ and $r > 0$ ($T = \infty$).

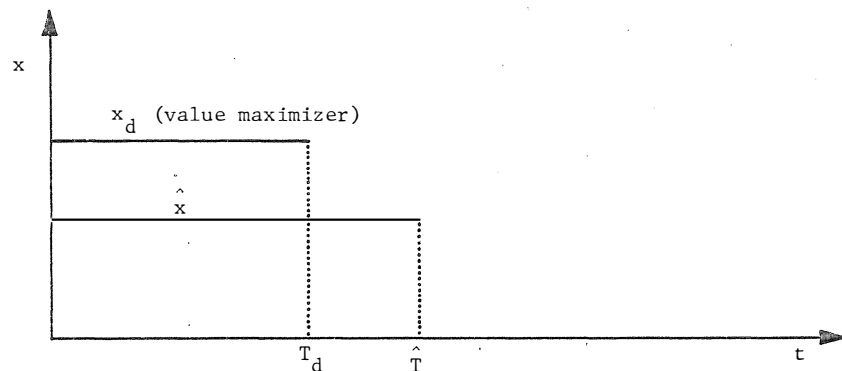


Figure III.1(ii): Value and utility maximizing paths when $F > 0$ and $r = 0$.

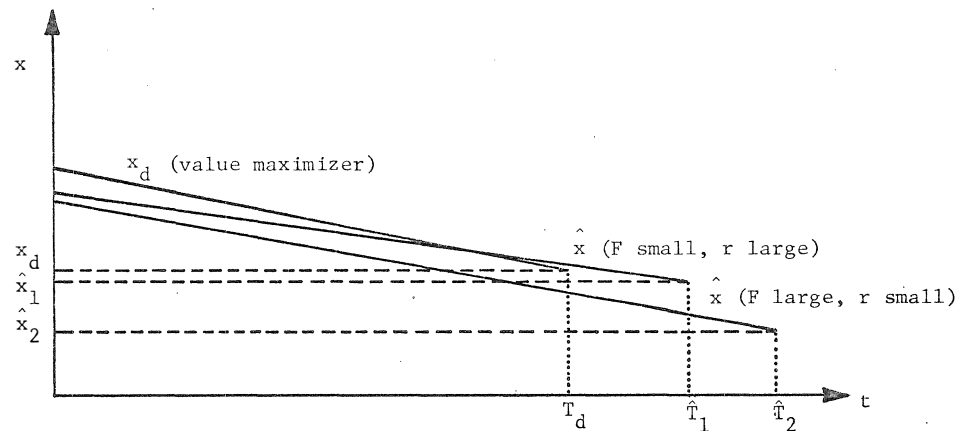


Figure III.1(iii): Value and utility maximizing paths when $F > 0$ and $r > 0$.

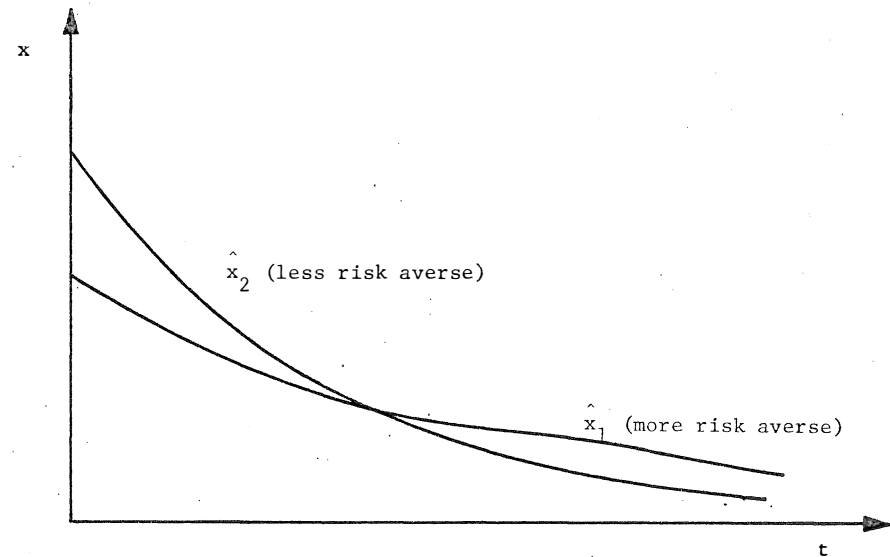


Figure III.1(iv): Effect of risk aversion on discounted utility maximizing path: $F = 0$, $r > 0$.

B. Franchise Tax

A franchise tax is a lump sum tax levied each period a firm remains in the industry; consequently it may be avoided in a dynamic framework through exit from the industry. As the tax tends to increase profit maximizing extraction rates (Proposition I.1) and simultaneously lower profits, one is tempted to infer that the utility maximizing firm's reaction is dependent upon the nature of risk aversion. However, the conclusions of Theorem III.2 hold for increasing or decreasing absolute (or relative) risk aversion.

Theorem III.2: Suppose that the producers of a non-renewable resource maximizes discounted utility of profits where the discount rate is positive. Then a franchise tax imposed at the rate B per time period hastens depletion; specifically if \hat{x} , \hat{T} and β, T_B represent the pre and post tax extraction rates and depletion dates, respectively, then $T_B < \hat{T}$ and $\beta(T_B) > \hat{x}(\hat{T})$. Moreover $\beta(t) > \hat{x}(t)$ for $t \in [0, T_B]$.

Corollary III.2: If the producer of a non-renewable resource is a utility maximizer, the discount rate is zero and fixed costs are positive, then the imperative of a franchise tax hastens depletion; specifically $\hat{x}(t) = \bar{x}$, a constant $\beta(t) = \bar{\beta}$, a constant and $\bar{\beta} > \bar{x}$.

Theorem III.2 and its corollary reinforce the deterministic result (Proposition II.1), indicating that concave utility causes no qualitative change in the reaction of an exhaustive firm to a franchise tax. Figures III.1(i) and III.2(ii) depict the franchise tax results.

C. Severance Tax

In practice, severance taxes are usually of the ad valorem type, but the analysis is presented in terms of the unit type for simplicity. The results are qualitatively the same for either case.

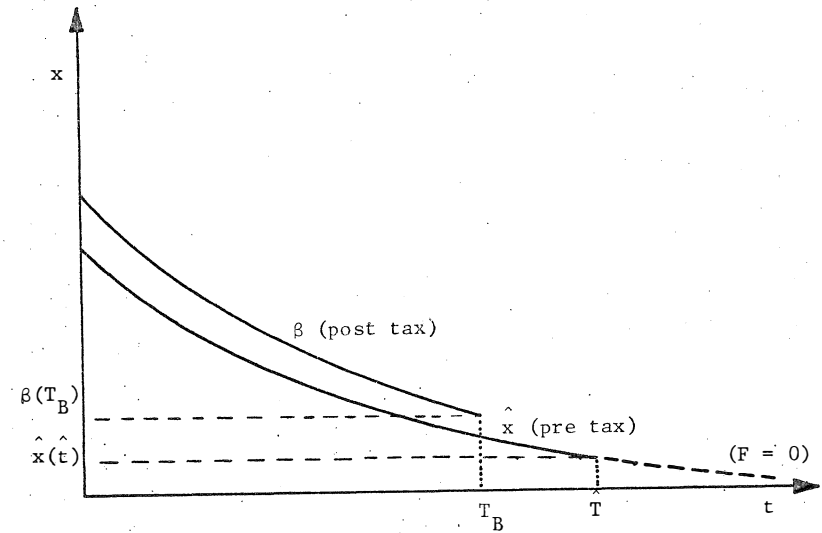


Figure III.2(i): Utility maximizing pre and post franchise tax extraction paths: $F \geq 0$ and $r > 0$ ($\hat{T} = \infty$ when $F = 0$).

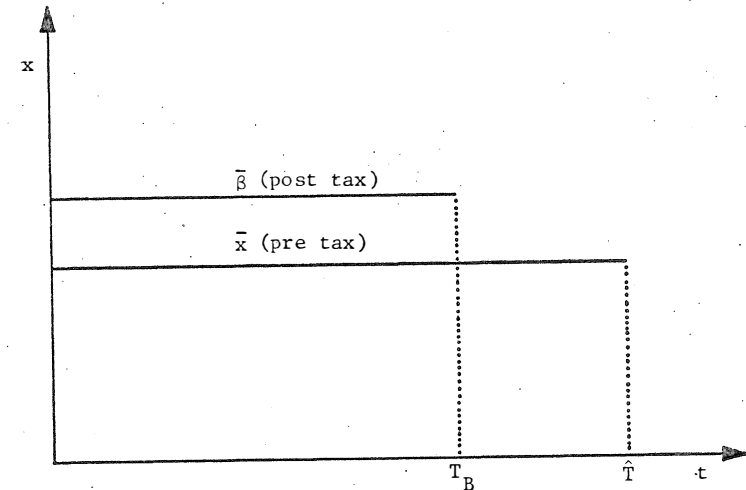


Figure III.2(ii): Utility maximizing pre and post franchise tax extraction paths: $F > 0$ and $r = 0$.

Proposition I.2 states that a severance tax will postpone depletion for a value maximizer for the case of zero fixed costs and a positive discount rate. The following theorem indicates that, under plausible conditions, a severance tax will have just the opposite effect on the decisions of a utility maximizer.

Theorem III.3a: Suppose that producers are utility maximizers and future utilities are discounted at a positive rate. In addition, suppose that fixed costs are zero and variable costs are linear. If absolute risk aversion is constant then the imposition of a severance tax at the rate A per unit of resource extracted results in hastened depletion; specifically, there exists a t^* such that $\alpha(t) > \hat{x}(t)$ for $t < t^*$ and $\alpha(t) < \hat{x}(t)$ for $t > t^*$, where \hat{x} and α are the pre and post tax depletion rates.

When fixed costs are positive and the discount rate is zero, Proposition I.2 asserts that a constant severance tax causes no change in optimal extraction plans. Since a severance tax reduces economic rents⁹, one would expect the effect of such a tax on the utility maximizers optimal plans would depend on his attitude toward risk. However, the following result is adduced independent of assumptions on measures of risk aversion.

Theorem III.3b: Suppose that producers are utility maximizers, the discount rate is zero and fixed costs are positive. A severance tax results in increased output and advanced depletion; specifically,

⁹ For a value maximizer with positive fixed costs and zero discount rate, the firm's supply curve is vertical with its base at the minimum of average cost.

$$\alpha(t) = \bar{\alpha} > \bar{x} = \hat{x}(t) \text{ for } t \in [0, T_A].$$

Figures III.3a and III.3b depict the results.

Theorems III.3a and III.3b and Proposition I.2 tend to challenge one rationale for a severance tax. Traditionally, it is argued that, due to inelasticity of demand for exhaustible resources, the burden of a severance tax is shifted primarily to the consumer; to the extent that the resource is exported, the tax is also. Present results contradict this argument. Proposition I.2 indicates no change in output in the deterministic case, and consequently the tax is borne through reduction in economic rents. In the face of producer risk aversion, results are even more striking as output rises. In either case, the tax is borne by the industry, at least in the short run.

D. Profit Tax

Proposition I.3 states that a value maximizer with zero fixed costs and a positive discount rate postpones production when faced with a constant profits tax. The following theorem shows that utility maximizer with constant absolute risk aversion will accelerate depletion.

Theorem III.4a: Suppose that producers are utility maximizers and that fixed costs are zero and the discount rate is positive. If absolute risk aversion is constant¹⁰, the imposition of a profits tax at the rate D per dollar at profits at a given time results in hastened depletion; i.e., there exists a t^* such that $\delta(t) < \hat{x}(t)$ for $t < t^*$ and $\delta(t) > \hat{x}(t)$ for $t > t^*$ where \hat{x} and δ are the pre and post tax extraction paths.

¹⁰ Both this theorem and Theorem III.3a hold for $\partial R_A / \partial \pi > 0$ as well; however this would be at least a tenuous assumption and hence it is not made.

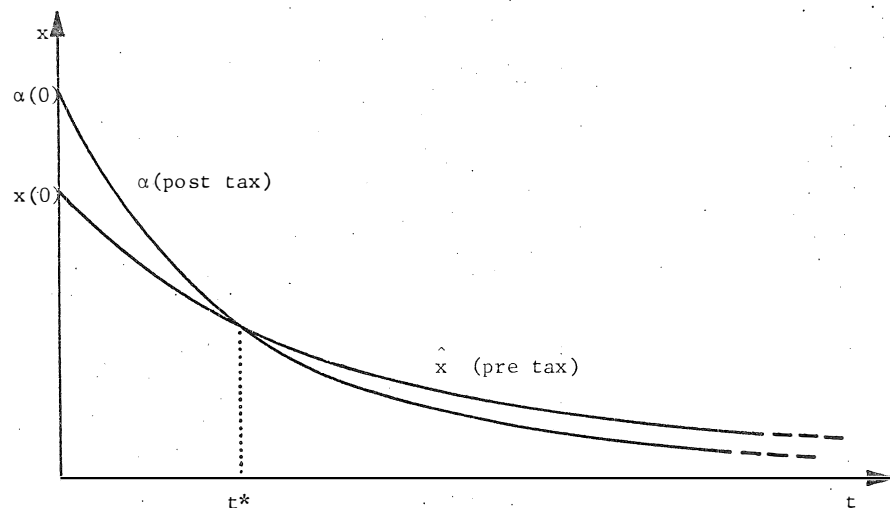


Figure III.3a: Utility maximizing pre and post severance tax extraction paths: $F = 0$ and $r > 0$.

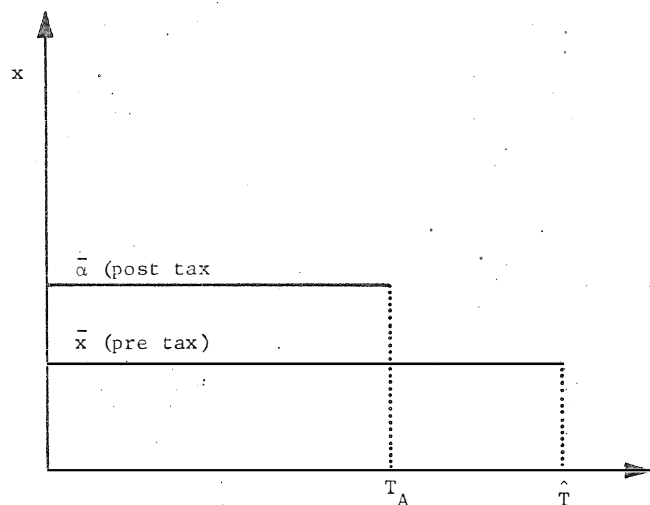


Figure III.3b: Utility maximizing pre and post severance tax extraction paths: $F > 0$ and $r = 0$.

Proposition I.3 also asserts that a value maximizer with positive fixed costs and a zero discount rate does not alter extraction plans in the face of a constant profits tax. We find below that the utility maximizer's reaction depends on quantitative aspects of attitudes toward risk.

Theorem III.4b: Suppose that producers are utility maximizers, fixed costs are positive and the discount rate is zero. Let the income elasticity of utility, $E_I(\pi) = d \ln u(\pi) / d \ln \pi$, and the income elasticity of marginal utility (measure of relative risk aversion), $R_R(\pi) = -d \ln u'(\pi) / d \ln \pi$, satisfy

$$(III.23) \quad E_I(\gamma\pi) + R_R(\gamma\pi) \leq 1$$

for $\gamma \in [0, 1]$. Then the imposition of a profit tax results in postponed depletion¹¹; in particular, since $r = 0$, $\hat{x}(t) = \bar{x}$ and $\delta(t) = \bar{\delta}$, and $\bar{\delta} < \bar{x}$.

These two results are depicted in Figures III.4a and III.4b, respectively. When fixed costs are present and $r > 0$ one can only conjecture the effect of a profit tax on exhaustion plans. Given the stated assumptions on the nature of risk aversion, it seems that generally extraction is postponed, but the exact manner in which this occurs depends the relative magnitudes of the discount rate and fixed costs, as Theorems III.4a and III.4b suggest.

¹¹Note that if $E_I(\pi) + R_R(\pi) \geq 1$, the opposite result holds.

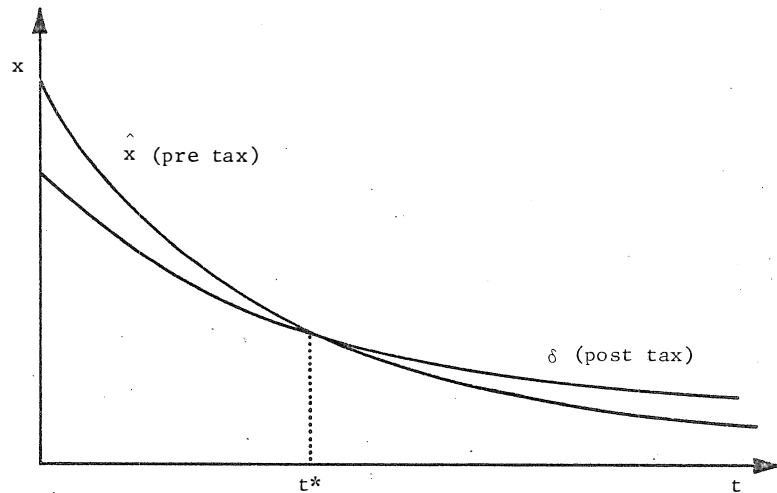


Figure III.4a: Utility maximizing pre and post tax extraction plans for a profit tax: $F = 0$ and $r > 0$.

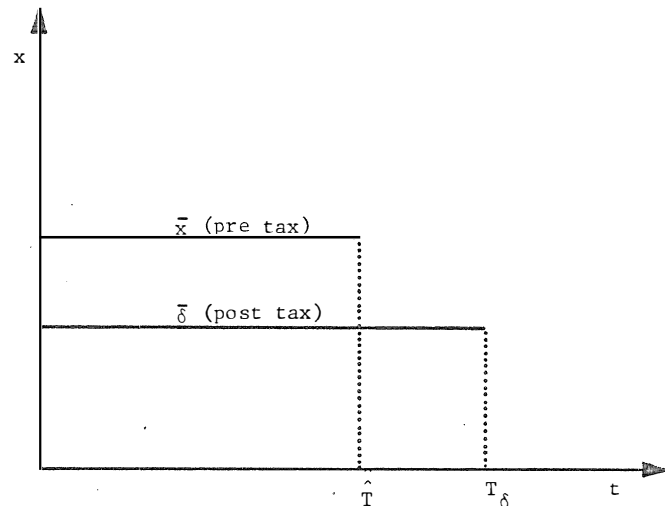


Figure III.4b: Utility maximizing pre and post tax extraction plans for a profit tax: $F > 0$ and $r = 0$.

IV. Risk Aversion

A. Risk-Averse Output

While the presence of risk aversion as due to price uncertainty introduce some ambiguities into the analysis, to some extent they can be resolved; price uncertainty causes no substantive alternation of the results obtained for concave utility alone. In contrast, the firm facing no resource constraint alters optimal behavior only after both concave utility and price uncertainty are present. In this section we examine the manner in which previous results generalize when price uncertainty is present.

Price uncertainty enters as an additive stochastic term appended to the demand function; the stochastic term has distribution function Φ . Deterministic price, p_d , is related to expected price, $E_\Phi(p)$, by the rule, $E_\Phi(p) \equiv E(p) = p_d$. Alternatively one could view the deterministic value maximizer as a risk-neutral expected utility maximizer; one can quickly verify that the behavior of both is identical.

When producers are risk-averse, they maximize the expected utility of profits¹²

$$(IV.1) \quad E(W) = \int_0^T e^{-rt} E[u(\pi)] dt$$

subject to (I.2). Necessary conditions require that along the optimal arc, x_s ,

$$(IV.2) \quad \lambda = e^{-rt} E[u'(\pi)(p-C')]$$

where λ is the constant shadow price of the resource, and that for $t = T$,

$$(IV.3) \quad \{E[u(\pi)] - E[u'(\pi)(p-C')]x\} e^{-rt} = 0$$

¹²We must assume that storage of extracted resources is not possible. See Leurs [9] for an analysis of extraction rules when storage is permitted in a deterministic setting.

Theorem IV.1a: Suppose that producers are risk-averse expected utility maximizers where the discount rate is positive and fixed costs are zero. If $g(\pi) = \pi u'(\pi)$ is convex, then the risk-averse expected utility maximizer postpones depletion vis-a-vis the value maximizer (or, alternatively, the risk-neutral expected utility maximizer); specifically, there exists a t^* such that $x_s(t) < x_d(t)$ for $t < t^*$ and $x_s(t) > x_d(t)$ for $t > t^*$.

Theorem IV.1b: Suppose producers are risk-averse expected utility maximizers and that the discount rate is zero while fixed costs are positive. The risk-averse expected utility maximizer postpones depletion vis-a-vis the value maximizer; specifically, $x_s(t) = \bar{x}_s$, $x_d(t) = \bar{x}_d$ and $\bar{x}_s < \bar{x}_d$. Moreover, if $g(\pi) = \pi u'(\pi)$ is convex, then $\bar{x}_s \in [\bar{x}, \bar{x}_d]$ where $\bar{x} = x(t)$.

Figures IV.1a and IV.1b depict these results. As in Theorem III.1, if fixed costs are positive and the discount rate is positive, the intersection of x_s and x_d cannot be unambiguously determined. Intersection or non-intersection is dependent on the relative magnitudes of fixed costs and the discount rate. However if intersection occurs, x_d intersects x_s from above.

Corollary IV.1: When $F > 0$ and $r = 0$, decreasing absolute and relative risk aversion imply that $\bar{x}_s \in [\bar{x}, \bar{x}_d]$.

The corollary notwithstanding, $\hat{x}_s \in [\hat{x}, x_d]$ is compatible with increasing relative risk-aversion as well, which is reassuring since a strong case for either increasing or decreasing relative-risk aversion cannot be established. These results are somewhat counter-intuitive since risk aversion might be expected to diminish output unambiguously, as in the competitive case. However, there is no clear parallel to the no-resource-constraint results, as price uncertainty and concave utility

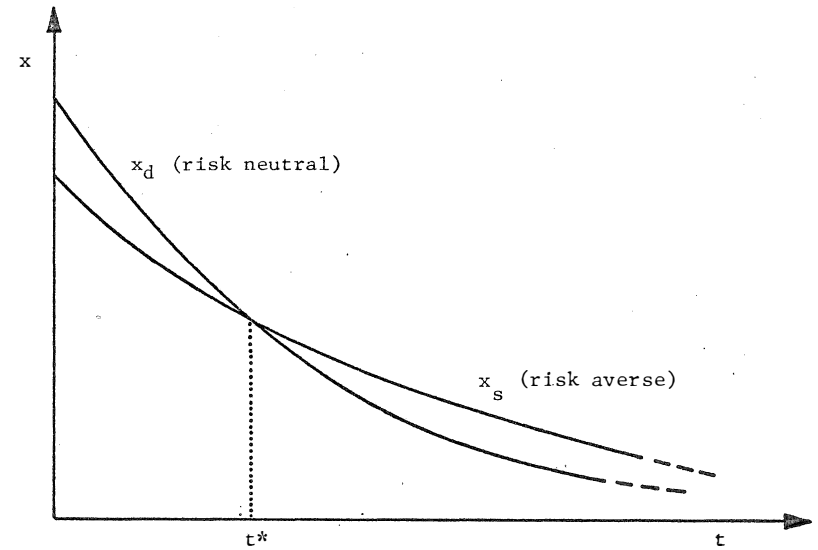


Figure IV.1a: Risk-averse and risk neutral paths:
 $F = 0$ and $r > 0$.

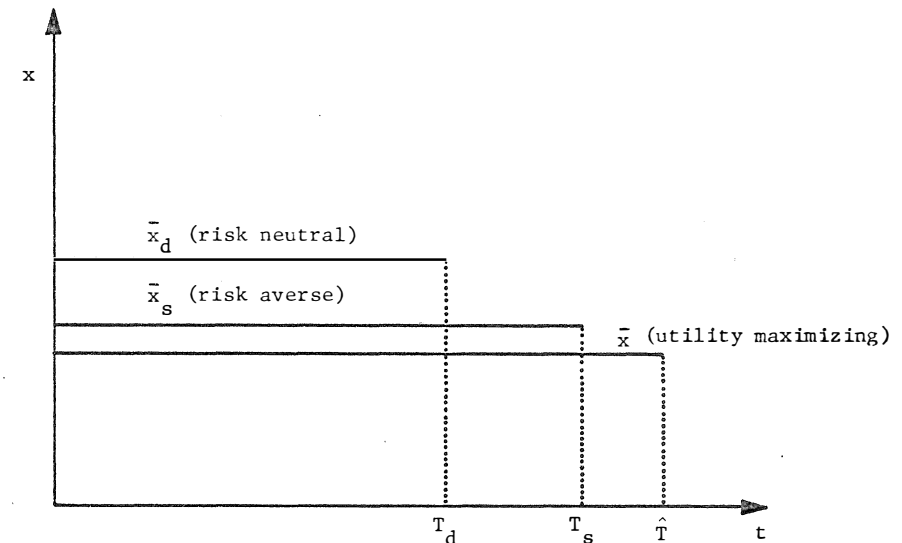


Figure IV.1b: Risk-averse, value, and utility maximizing paths: $F > 0$ and $r = 0$.

are non-trivially separable in the exhaustive case. Nevertheless, it is enlightening to distinguish the separate effects of concave utility and price uncertainty (risk aversion) on an exhaustive firm.

B. Franchise Tax

Under risk aversion the franchise tax results generalize in the following manner.

Theorem IV.2: Suppose that resource price is stochastic and producers are risk-averse expected utility maximizers, where fixed costs are non-negative and the discount rate is positive. Moreover, suppose that the distribution of price is such that $p > C'(x^*)$ where x^* satisfies $x^*C'(x^*) = C(x^*)$. Then a franchise tax, B per time period, results in hastened depletion; specifically if x_s , T_s and β_s , T_B are the pre and post tax extraction rates and depletion dates, then $T_B < T_s$, and $\beta(T_B) > x_s(T_s)$ and $\beta(t) > x_s(t)$ for $t \in [0, T_B]$.

Corollary IV.2: Suppose that the hypothesis of Theorem IV.2 holds with the exception that $F > 0$ and $r = 0$. Then $x_s(t) = \bar{x}_s$, $\beta_s(t) = \bar{\beta}_s$ and $\bar{\beta}_s > \bar{x}_s$.

Figures IV.2a and IV.2b depict these results.

C. Severance Tax

Theorem III.3a does not generalize directly when risk aversion due to price uncertainty is present. However Theorem III.3b can be extended in the following manner.

Theorem IV.3(i): Suppose that producers are risk-averse expected utility maximizers where fixed costs are positive and the discount rate is zero.

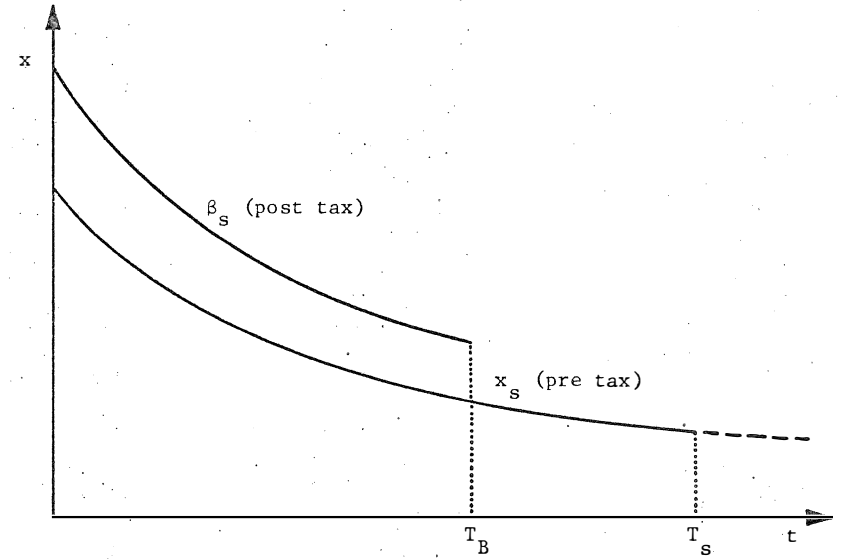


Figure IV.2a: Risk-averse pre and post franchise tax extraction paths: $F \geq 0$ and $r > 0$ ($T_s = \infty$ when $F = 0$).

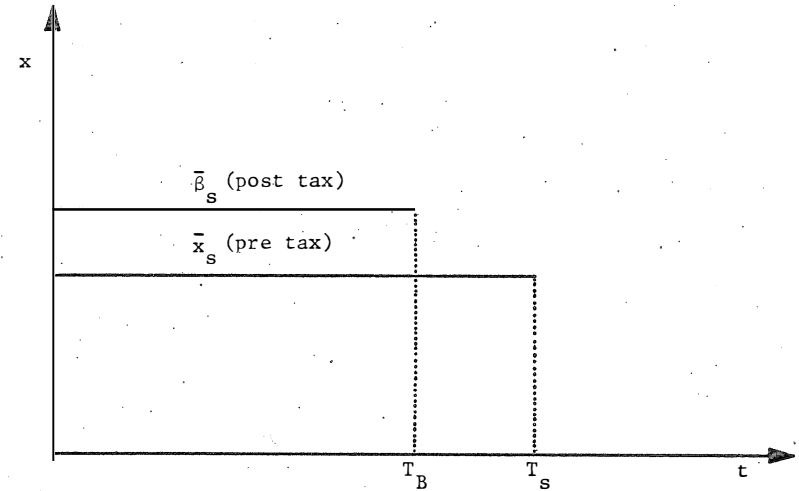


Figure IV.2b: Risk-averse pre and post franchise tax extraction paths: $F > 0$ and $r = 0$.

If a severance tax is imposed at the rate A per unit of output extracted, and the distribution of price is such that $p > C'(x^*) + A$, where $x^*C'(x^*) = C(x^*)$, then depletion is hastened; in particular, $x_s(t) = \bar{x}_s$ and $\alpha_s(t) > \bar{\alpha}_s$ where x_s and α_s are the pre and post tax extraction paths and $\bar{\alpha}_s > \bar{x}_s$.

Theorem IV.3(ii) presents a local result analogous to that of Theorem IV.3(i) without restricting the distribution of price.

Theorem IV.3(ii): Suppose that producers are risk-averse expected utility maximizers where fixed costs are positive and the discount rate is zero.

If A is a severance tax rate then $\partial x_s / \partial A > 0$.

Figure IV.3 depicts these results;

D. Profit Tax

As with Theorem III.3a, Theorem IV.3a does not generalize. However Theorem III.4b can be extended in the following manner.

Theorem IV.4: Suppose that in the presence of price uncertainty producers are risk-averse expected utility maximizers where fixed costs are positive and the discount rate is zero. If $E_I(\pi) + R_R(\pi) \leq 1$ and the distribution of price is such that $p > C'(x^*)$ where $x^*C'(x^*) = C(x^*)$, then the imposition of a profits tax at the rate D results in postponed depletion; specifically, $x_s(t) = \bar{x}_s$, $\delta_s(t) = \bar{\delta}_s$ where \bar{x}_s and $\bar{\delta}_s$ are the pre and post tax depletion rates and $\bar{\delta}_s < \bar{x}_s$.

Figure IV.4 depicts this result. However we should be quick to note that $E_I(\pi) + R_R(\pi) \geq 1$ implies the opposite result, namely, $\bar{\delta}_s > \bar{x}_s$. While Arrow [1] argues for certain limiting values of the measure of relative risk aversion, little can be said about its value at intermediate

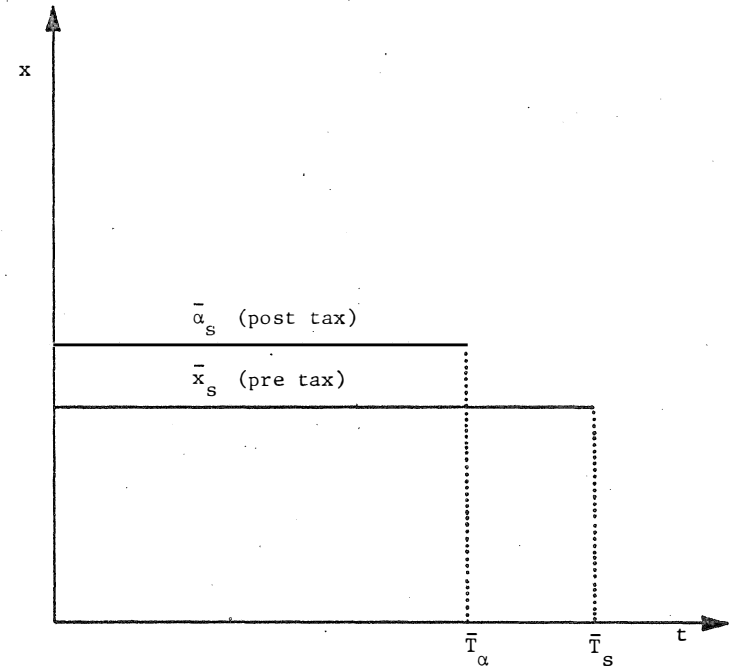


Figure IV.3: Risk-averse pre and post severance tax extraction paths: $F > 0$ and $r = 0$.

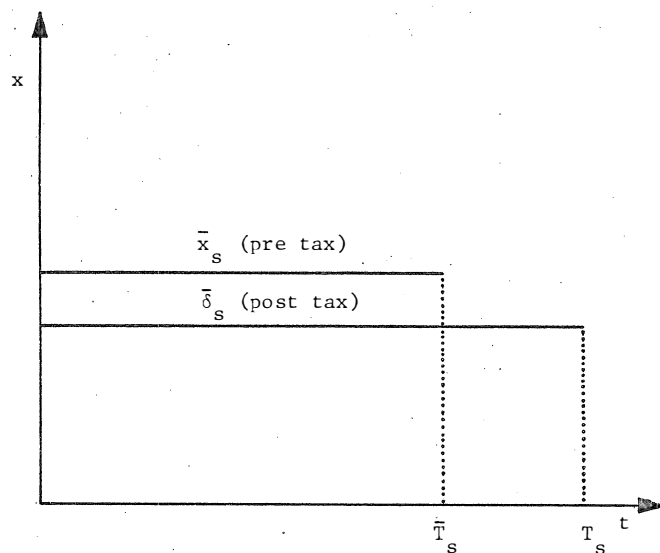


Figure IV.4: Risk-averse pre and post profit tax extraction paths: $F > 0$ and $r = 0$ ($E_I(\pi) + R_R(\pi) \leq 1$).

incomes. Consequently the role of a profit tax is difficult to assess here as well as when price uncertainty is absent.

V. Summary of Results and Conclusion

Generally risk aversion tends to postpone production plans for an extractive firm. However the actual configuration of optimal path under risk-aversion vis-a-vis a value maximizing plan depends on assumptions concerning fixed costs (F) and the discount rate (r). When $F > 0$ and $r = 0$, the value maximizing plan exceeds the utility maximizing plan until the former plan results in resource depletion (Figure III.1(ii)). When $F = 0$ and $r > 0$, the value maximizing plan is larger initially, but the utility maximizing plan is larger terminally (Figure III.1(i)). Clearly then when both $F > 0$ and $r > 0$, the intersection or non-intersection of the two paths will depend on the relative magnitudes of F and r . (Figure III.1(iii)). One would expect that when fixed costs are large relative to the discount rate, the schedules would not intersect and vice-versa. The postponement of depletion caused by risk aversion is intuitively plausible: risk aversion causes an evening out of the production schedule, or, alternatively, a spreading of more equal incomes over a longer horizon.

Figure V.1 summarizes the effects of constant franchise, severance and profit taxes under a variety of scenarios. Generally these results speak for themselves, but several interesting observations can be made. First, when no resource constraint is present, risk aversion alters the standard tax results in an intuitively plausible way under plausible

	I		II	III	
Firm Type	F & r	Franchise Tax ($\dot{B} = 0$)	Severance Tax ($\dot{A} = 0$)	Profit Tax ($\dot{D} = 0$)	
No Resource Constraint: Value Maximizer	$F \geq 0$ $r \geq 0$	No change	Reduced production	No change	A
No Resource Constraint: Expected Utility Maximizer	$F \geq 0$ $r \geq 0$	Reduced production $\left(\frac{\partial R_A(\pi)}{\partial \pi} < 0\right)$	Reduced production $\left(\frac{\partial R_A(\pi)}{\partial \pi} < 0\right)$	Increased or reduced production as $\frac{\partial R_R(\pi)}{\partial \pi} \geq 0$	B
Extractive Firm: Value Maximizer	$F > 0$ $r = 0$	Depletion accelerated	No change	No change	C
	$F \geq 0$ $r > 0$	Depletion accelerated	Depletion postponed	Depletion postponed	D
Extractive Firm: Utility Maximizer	$F > 0$ $r = 0$	Depletion accelerated	Depletion accelerated	Depletion postponed or accelerated as $E_I(\pi) + R_R(\pi) \leq 1$	E
	$F > 0$ $r > 0$	Depletion accelerated ($F \geq 0$)	Depletion accelerated $\left(\frac{\partial R_A(\pi)}{\partial \pi} = 0\right)$	Depletion postponed $\left(\frac{\partial R_A(\pi)}{\partial \pi} = 0\right)$	F
Extractive Firm: Expected Utility Maximizer	$F > 0$ $r = 0$	Depletion accelerated	Depletion accelerated	Depletion postponed or accelerated as $E_I(\pi) + R_R(\pi) \leq 1$	G
	$F = 0$ $r > 0$	Depletion accelerated	Undetermined	Undetermined	H

Figure V.1: Summary of Tax Results

economic hypotheses (rows A and B; these are the Sandmo [10] results). Second, when a resource constraint exists, the value maximizing results can be defended intuitively (rows C and D; these and related results are found in [2]). However, when an extractive firm maximizes utility or expected utility, intuition is no longer a reliable guide and ambiguities appear (rows E - H). In particular, comparing the element in row C and column II (denoted C-II) with E-II, a severance tax results in accelerated depletion when risk aversion exists (fixed costs being positive and the discount rate zero), but causes no change in the value maximizers output. When discounting occurs at a positive rate and fixed costs are zero, a severance tax causes accelerated depletion under risk aversion and postponed depletion for the value maximizer (compare D-II and F-II). For a profit tax when the discount rate is zero and there are positive fixed costs value maximizing path is unaltered, while under risk aversion depletion may be accelerated, postponed or unchanged (compare C-III with E-III or G-III). These incongruencies are not problematic in themselves but become so in light of the tenuous hypotheses concerning risk attitudes under which the risk aversion results are derived.

Also crucial is the explicit assumption of imperfect capital markets. In order that the effect of various tax policies on the production decisions of extractive firms might be anticipated, perhaps a necessary precondition for policy would be firms' accessibility to capital markets. Under these conditions firms would just be value maximizers and achieve their desired income streams through the capital market; when firms are value maximizers there is little trouble in assessing their probable reaction to taxes and subsidies. In the coal industry, for example,

many firms, especially smaller ones, are reliant primarily on equity financing and the consequent lack of capital availability has frequently been singled out as a deterrent to achieving adequate production levels. This is certainly in keeping with the results presented here. Proposed programs to give entering firms better access to debt markets would promise to ameliorate the situation.

APPENDIX

Proof of Theorem III.1: We prove only parts (i) and (iii) as (ii)

follows directly from the proof of (iii).

(i) We first know that the utility maximizers depletion date is infinite. To do this suppose \hat{T} is finite. Transversality conditions require that

$$(A.1) \quad u'(\pi) [p - C'(x)]x = u(\pi)$$

which in view of the neoclassical assumption and the concavity of u implies that $\hat{x}(\hat{T}) = 0$. However $\hat{x}(\hat{T}) = 0$ and \hat{T} finite violates the necessary conditions (II.4). Hence $T = \infty$ and, by the resource constraint, $\hat{x}(\hat{T}) = 0$. If the cost function is neoclassical (i.e., $\lim_{z \rightarrow \infty} C'(z) = 0$ and $\lim_{z \rightarrow 0} C'(z) = \infty$) a similar technique shows that $T_d = \infty$ and $x_d(T_d) = 0$ as well.

Differentiating the necessary conditions (I.3) and (II.4) with respect to t yields, respectively,

$$(A.2) \quad \dot{x}_d = \frac{-r [p - C'(x)]}{C''(x)}$$

and

$$(A.3) \quad \dot{\hat{x}} = \frac{-r[p - C'(x)]}{R_A(\pi) [p - C'(x)]^2 + C''(x)}$$

where $R_A(\pi)$ is the measure of absolute risk aversion. For any t^* such that $\hat{x}(t^*) = x_d(t^*)$

$$(A.4) \quad \dot{x}_d(t^*) - \dot{\hat{x}}(t^*) = \frac{-r[p - C'(x)]^3 R_A(\pi)}{C''(x) [R_A(\pi) (p - C'(x))^2 + C''(x)]} < 0$$

In view of the resource constraint and $\hat{T} = T_d = \infty$ such a t^* must exist and be unique, thus completing the proof of (i).

(iii) In view of the neoclassical assumption and the presence of fixed costs it is clear that \hat{T} and T_d are finite. Transversality conditions for the utility maximizers are, for $x = \hat{x}(\hat{T})$,

$$(A.5) \quad u'(\pi) [p - C'(x)]x = u(\pi)$$

Subtracting $u'(\pi)\pi$ from each side and dividing by $u'(\pi)$ yields

$$(A.6) \quad C(x) - xC'(x) = \frac{u(\pi)}{u'(\pi)} - \pi$$

Since u is concave, $u'(\pi) < u(\pi)/\pi$, hence both sides of (A.6) are positive so that $\hat{x}(\hat{T}) < x_d(T_d)$. Conjoined with (A.4) this implies that $T > T_d$. Note however that we are not assured of the existence of a point such as t^* , but only that if such a point exists, x_d intersects \hat{x} from above.

Proof of Corollary III.1: The proof is immediate in view of the resource constraint, the infinite depletion date, and equation (A.3).

Proof of Theorem III.2: We first show that $\hat{x}(\hat{T}) < \beta(T_B)$. If $\pi = px - C(x)$ and $B > 0$ then

$$(A.7) \quad \frac{u(\pi - B)}{x} < \frac{u(\pi)}{x}$$

for $x = \hat{x}(\hat{T})$. Also it is clear that

$$(A.8) \quad u'(\pi - B) [p - C'(x)] \geq u'(\pi) [p - C'(x)]$$

for $x = \hat{x}(\hat{T})$. When the tax is imposed,

$$(A.9) \quad u'(\pi - B) [p - C'(x)] - \frac{u(\pi - B)}{x} = 0.$$

for $x = \beta(T_B)$. In view of (A.7) and (A.8)

$$(A.10) \quad u'(\pi - B) [p - C'(x)] - \frac{u(\pi - B)}{x} \geq 0.$$

for $x = \hat{x}(\hat{T})$. Denoting the left hand side of (A.10) as $H(x)$, we also see that

$$(A.11) \quad H'(x) = u''(\pi - B) [p - C'(x)]^2 - u'(\pi - B) C''(x) - H(x)/x \leq 0$$

for $H(x) \geq 0$. Since $H(x) = 0$ implies $x = \beta(T_B)$ by (A.9), clearly $\beta(T_B) > \hat{x}(T)$.

We now show that \hat{x} and β do not intersect. Suppose to the contrary that there exists a t^* such that $\hat{x}(t^*) = \beta(t^*)$. If λ and λ_B are the pre and post tax multipliers associated with the resource constraint then

$$(A.12) \quad \lambda_B(t^*) = e^{-rt} u'(\pi-B) [p-C'(x)]|_{t=t^*} > e^{-rt} u'(\pi) [p-C'(x)]|_{t=t^*} = \lambda(t^*)$$

Differentiating the transversality condition (A.9) with respect to B and solving for $\partial x / \partial B$ yields

$$(A.13) \quad \frac{\partial x}{\partial B} = \frac{u''(\pi-B)(p-C')x - u'(\pi-B)}{u''(\pi-B)(p-C')^2 x - C''u'(\pi-B)x}$$

Differentiating the left side of (A.12) with respect to B yields

$$(A.14) \quad \frac{d\lambda}{dB} = e^{-rt} \left\{ u''(\pi-B)(p-c') \left[(p-c') \frac{\partial x}{\partial B} - 1 \right] - u'(\pi-B) C'' \frac{\partial x}{\partial B} \right\}$$

Substituting from (A.13) yields, for $t=T_B$,

$$(A.15) \quad \frac{d\lambda}{dB} = e^{-rT_B} \left\{ \frac{-u'(\pi-B)}{x} \right\} < 0$$

which contradicts (A.12). Hence $\beta(t) > \hat{x}(t)$ for $t \in [0, T_B]$. In view of the resource constraint this implies $T_B < \hat{T}$ and the proof is complete.

Proof of Theorem III.3a: Since fixed costs are zero, the post tax depletion date is infinite by a technique similar to that used in the first part of the proof of Theorem III.1. Necessary conditions for the pre and post tax problems are

$$(A.16) \quad \lambda = e^{-rt} u'(\pi) [p-C'(x)]$$

and

$$(A.17) \quad \lambda_A = e^{-rt} u'(\pi-Ax) [p-C'(x) - A]$$

Differentiating these two expressions with respect to time, recalling that

$\dot{\lambda} = \dot{\lambda}_A = 0$, and solving for \dot{x} and $\dot{\alpha}$, where \hat{x} and α are the pre and post

tax optimal paths, yields

$$(A.18) \quad \dot{x} = \frac{-r(p-C')}{R_A(\pi)(p-C')^2 + C''}$$

and

$$(A.19) \quad \dot{\alpha} = \frac{-r(p-C'-A)}{R_A(\pi-Ax)(p-C'-A)^2 + C''}$$

In view of the resource constraint and the infinite depletion date, \hat{x} and α must intersect at some point t^* . The linearity of $C(x)$ implies that the expression for $\dot{x} - \dot{\alpha}$ at t^* is

$$(A.20) \quad \dot{x} - \dot{\alpha} = -r(p-C')(p-C'-A)[R_A(\pi-Ax)(p-C'-A) - R_A(\pi)(p-C')]/D$$

where $D = R_A(\pi-Ax)R_A(\pi)(p-C'-A)^2(p-C')^2 > 0$. Since $R_A(\pi-Ax) = R_A(\pi) = R_A^*$, (A.20) becomes

$$(A.21) \quad \dot{x} - \dot{\alpha} = -r(p-C')(p-C'-A)(-R_A^*)A/D > 0$$

Hence \hat{x} intersects α from below, and the theorem is proved.

Proof of Theorem III.3b: Observe that for $\pi > 0$

$$(A.22) \quad d \left[\frac{du'(\pi)/dx}{u(\pi)} \right] = \frac{u(\pi) u''(\pi) [p-C'(x)] - u'(\pi)^2}{u(\pi)^2} \leq 0$$

Since $\pi > \pi - Ax$, where A is the tax rate, (A.22) implies

$$(A.23) \quad \frac{u'(\pi) [p-C'(x)]}{u(\pi)} \leq \frac{u'(\pi-Ax) [p-C'(x)-A]}{u(\pi-Ax)}$$

for $x = \bar{x}$, where \bar{x} is the pre tax optimal output rate. Letting

$a = u(\pi-Ax)/[u(\pi)]$ when $x = \bar{x}$, (A.23) can be rewritten as

$$(A.24) \quad au'(\pi) [p-C'(x)] < u'(\pi-Ax) [p-C'(x)-A]$$

for $x = \bar{x}$. In addition

$$(A.25) \quad \frac{au(\pi)}{x} = \frac{u(\pi-Ax)}{x}$$

at $x = \bar{x}$. In view of the necessary conditions, equations (A.24) and

(A.25) imply

$$(A.26) \quad G(x) = u'(\pi - Ax) [p - C'(x) - A] - \frac{u(\pi - Ax)}{x} \geq 0$$

at $x = \bar{x}$. If \bar{a} is the optimal extraction rate after the imposition of the tax, then $G(\bar{a}) = 0$. This, together with the fact that

$$(A.27) \quad \frac{dG}{dx} = u''(\pi - Ax) [p - C'(x) - A]^2 - u'(\pi - Ax) C''(x) - \frac{G(x)}{x} \leq 0$$

for $G(x) \geq 0$ implies that $\bar{a} \geq \bar{x}$.

Proof of Theorem III.4a: Since fixed costs are zero both pre and post tax depletion dates are infinite. By the resource constraint there exists a t^* such that \hat{x} and δ intersect at t^* . Note that

$$(A.28) \quad \dot{\delta} = \frac{-r(p - C')}{R_A(\gamma\pi)\gamma(p - C')^2 + C''}$$

Where $\gamma = 1 - D$, so that

$$(A.29) \quad \frac{\dot{\delta}}{\delta} = \frac{-r(p - C')^3 [R_A(\gamma\pi)\gamma - R_A(\pi)]}{\Delta}$$

Where $\Delta = [R_A(\pi)(p - C')^2 + C''] [R_A(\gamma\pi)\gamma(p - C')^2 + C''] > 0$

Since $R_A(\gamma\pi) = R_A(\pi) = R_A^*$, (A.29) becomes

$$(A.30) \quad \frac{\dot{\delta}}{\delta} - \frac{\dot{\hat{x}}}{\hat{x}} = -r(p - C')^3 R_A^* (\gamma - 1) / \Delta > 0, \text{ which yields the desired result.}$$

Proof of Theorem III.4.b: The proof is aided by use of the following lemma.

Lemma: Suppose that $u(\pi)$ is continuously differentiable with $u'(\pi) > 0$ and $u''(\pi) \leq 0$. If the income elasticity of utility, $E_I(\pi) = d \ln u(\pi) / d \ln \pi$, and the income elasticity of marginal utility (measure of relative risk aversion), $R_R(\pi) = -d \ln u'(\pi) / d \ln \pi$, satisfy

$$(A.31) \quad E_I(\gamma\pi) + R_R(\gamma\pi) \leq 1$$

for $\gamma \in [0, 1]$ then

$$(A.32) \quad u(\gamma\pi^*) u(\pi) \leq u(\gamma\pi) u(\pi^*)$$

for $\pi \in [0, \pi^*]$.

We continue now with the proof of the theorem. Let $\pi = px - C(x)$, $\gamma = 1 - D$, where D is the tax rate, and $\bar{\delta}$ is the value of x such that $\int_0^T u(\gamma\pi) dt$ is maximized subject to (I.2). Define $b = [u(\gamma\pi)/u(\pi)]|_{x=\bar{x}}$. Since u is unique up to a linear monotonic transformation, the maximum of $\int_0^T bu(\pi) dt$ subject to (I.2) still occurs at \bar{x} . Clearly

$$(A.33) \quad \frac{u(\gamma\pi)}{x} = \frac{bu(\pi)}{x}$$

for $x = \bar{x}$. By the Lemma

$$(A.34) \quad b u(\pi) \leq u(\gamma\pi)$$

for $\pi \in [0, \pi(\bar{x})]$. Consequently

$$(A.35) \quad \frac{du(\gamma\pi)}{d\pi} \leq b \frac{du(\pi)}{d\pi}$$

for π close to $\pi(\bar{x})$. To see this suppose there exists an $\eta > 0$ such that

$$(A.36) \quad \frac{du(\gamma\pi)}{d\pi} > b \frac{du(\pi)}{d\pi}$$

for $\pi \in [\bar{\pi} - \eta, \bar{\pi}]$, where $\bar{\pi} = \pi(\bar{x})$. Integrating both sides of (A.36) between $\bar{\pi} - \eta$ and $\bar{\pi}$ and evaluating in light of the definition of b yields

$$(A.37) \quad u(\gamma\pi)|_{\bar{\pi}-\eta} < b u(\pi)|_{\bar{\pi}-\eta}$$

in contradiction to (A.34). Hence (A.35) holds.

In view of (A.33), (A.35), and the transversality conditions (III.1)

$$(A.38) \quad F(x) = u'(\gamma\pi) [p - C'(x)] - \frac{u(\gamma\pi)}{x} \leq 0$$

for $x = \bar{x}$. Since $F(\delta) = 0$ and

$$(A.39) \quad F'(x) = u''(\gamma\pi)\gamma[p - C'(x)]^2 - u'(\gamma\pi)C''(x) - \frac{F(x)}{x} \leq 0$$

in order that \bar{x} and $\bar{\delta}$ yield maxima, then it follows that $\bar{\delta} < \bar{x}$ as desired.

Proof of Lemma: Expression (A.31) can be manipulated and rewritten as

$$(A.40) \quad \frac{u(\gamma\pi) [\gamma\pi u''(\gamma\pi) + u'(\gamma\pi)] - \gamma\pi u'(\gamma\pi)^2}{u(\gamma\pi)^2} = \frac{d \left[\frac{u'(\gamma\pi)}{u(\gamma\pi)} \right]}{d\gamma} \geq 0$$

for $\pi > 0$ and $\gamma \in [0,1]$. Consequently

$$(A.41) \quad \frac{\gamma u'(\gamma\pi)}{u(\gamma\pi)} \leq \frac{u'(\pi)}{u(\pi)}.$$

Equation (A.41) can be rewritten as

$$(A.42) \quad \frac{\gamma\pi u'(\gamma\pi)u(\pi) - \pi u'(\pi)u(\gamma\pi)}{u(\pi)^2} = d \left[\frac{u(\gamma\pi)}{u(\pi)} \right] \leq 0$$

Hence for $\gamma \in [0,1]$ and $\pi \in [0, \pi^*]$

$$(A.43) \quad \frac{u(\gamma\pi^*)}{u(\pi^*)} \geq \frac{u(\gamma\pi)}{u(\pi)}.$$

Since (A.43) can be rewritten as (A.32) the proof is complete.

Proof of Theorem IV.1a: As in Theorem III.1(i), the depletion date is infinite when fixed costs are zero. Hence x_s is identical to x_d or there is at least one intersection. Suppose that $x_s(t^*) = x_d(t^*)$.

From the necessary conditions

$$(A.44) \quad \dot{x}_d = \frac{-r(p-C')}{C''}$$

and

$$(A.45) \quad \dot{x}_s = \frac{rE[u'(\pi)(p-C')]}{E[u''(\pi)(p-C')^2 - u'(\pi)C'']}$$

Hence

$$(A.46) \quad \dot{x}_d - \dot{x}_s = r \left\{ E(p)E[u'(\pi)] - E[pu'(\pi)] - E(p-C')E[u''(\pi)(p-C')^2] \right\} / \Delta$$

where $\Delta = C''E[u''(\pi)(p-C')^2 - u'(\pi)C''] < 0$

Observe that the first two terms in the numerator of (A.46) can be rewritten as $E(\pi) E[u'(\pi)] - E[\pi u'(\pi)]$.

However since $g(\pi) = \pi u'(\pi)$ is convex

$$(A.47) \quad E[g(\pi)] > g[E(\pi)]$$

by Jensen's inequality. Thus the numerator of (A.46) is positive so that (A.46) is negative.

Proof of Theorem IV.1b: Since fixed costs are positive depletion dates are finite; also, since $r = 0$, optimal rates are constant. Following the proof of Theorem III.1(ii) equation (IV.3) can be rewritten as

$$(A.48) \quad C(x) - xC'(x) = E[u(\pi) - \pi u'(\pi)] / E[u'(\pi)]$$

for $x = \bar{x}_s$. In view of the concavity of u , (A.48) is positive, establishing the first conclusion.

For the second part observe that $g(\pi)$ convex implies $u'(\pi)$ convex.

Thus by Jensen's inequality

$$(A.49) \quad E[u'(\pi)] \geq u'[E(\pi)]$$

for all $x \geq 0$. Likewise $g(\pi)$ convex implies $u(\pi) - g(\pi)$ concave so that

$$(A.50) \quad E[u(\pi) - \pi u'(\pi)] \leq u[E(\pi)] - E(\pi)u'[E(\pi)]$$

for all $x \geq 0$.

At the utility maximizing output, \bar{x} , (A.6) can be rewritten as

$$(A.51) \quad u'[E(\pi)] [C(x) - xC'(x)] = u[E(\pi)] - E(\pi)u'[E(\pi)]$$

where $E(p)$ is equal to deterministic price. In view of (A.49), (A.50),

and (A.51)

$$(A.52) \quad E[u'(\pi)] [C(x) - xC'(x)] \geq E[u(\pi) - u'(\pi)]$$

at \bar{x} , or alternatively,

$$(A.53) \quad E[u'(\pi)(p-C'(x))] E\left[\frac{u(\pi)}{x}\right] \geq 0$$

at \bar{x} . Since (A.53) is decreasing in x , $\bar{x}_s \geq \bar{x}$ concluding the proof.

Proof of Theorem IV.2: The proof follows the same format as Theorem III.2 with expected values replacing deterministic values and hence is omitted.

Proof of Corollary IV.2: The proof follows directly from that of Theorem IV.2.

Proof of Theorem IV.3(i): The proof is similar to that of Theorem III.3b.

Since $p > C'(x^*) + A$ and $x_s < x^*$, we observe that $p - C'(x_s) - A > 0$ and $\pi > 0$.

Hence (A.18) and (A.19) in the proof of Theorem III.3b still hold.

Letting

$$(A.54) \quad a(\pi) = \frac{u(\pi - Ax)}{u(\pi)}$$

when $x = \bar{x}_s$, we can rewrite (A.24) and (A.25) in view of (A.54). Taking expectations yields

$$(A.55) \quad E[au'(\pi) (p - C'(x))] \leq E[u'(\pi - Ax) (p - C'(x) - A)]$$

and

$$(A.56) \quad E\left[\frac{au(\pi)}{x}\right] = E\left[\frac{u(\pi - Ax)}{x}\right]$$

at \bar{x}_s . Similar to Theorem III.3b, (A.55), (A.56), and (IV.3) imply that

$$\bar{\alpha}_s \geq \bar{x}_s.$$

Proof of Theorem IV.3(ii): Conditions corresponding to (IV.3) imply that

$\bar{\alpha}_s$ satisfies

$$(A.57) \quad E[u'(\pi - Ax) (p - C'(x) - A)] = E\left[\frac{u(\pi - Ax)}{x}\right]$$

Differentiating (A.57) totally with respect to A and solving for $\partial x / \partial A$ yields

$$(A.58) \quad \frac{\partial x}{\partial A} = \frac{E[u''(\pi - Ax)]}{E[\Delta]}$$

where $\Delta = u''(\pi - Ax) [p - C'(x) - A]^2 x - xC''(x)u'(\pi - Ax)$. Clearly $\partial x / \partial A > 0$ and the theorem is proved.

Proof of Theorem IV.4: The proof is similar to that of Theorem III.4b.

For a particular value of π let $b(\pi)$ be the number such that

$$(A.59) \quad \frac{u(\gamma\pi)}{x} = \frac{b(\pi) u(\pi)}{x}$$

at $x = \bar{x}_s$, where $\gamma = (1-D)$ and D is the tax rate. By the lemma of Theorem III.4b and the proof of Theorem III.4b

$$(A.60) \quad \frac{du(\gamma\pi)}{d\pi} \leq b(\pi) \frac{du(\pi)}{d\pi}$$

for $x = \bar{x}_s$ and any value of p . In view of the non-negativity of $p - C'(x)$

and hence π , and in light of (IV.3), taking expectations over (A.59)

and (A.60) yields

$$(A.61) \quad E[u'(\gamma\pi) (p - C'(x))] \leq E\left[\frac{u(\gamma\pi)}{x}\right]$$

at $x = \bar{x}_s$. By analogy with Theorem III.4b, we conclude that $\bar{\delta}_s \leq \bar{x}_s$.

REFERENCES

- [1] Arrow, K. J., Essays in the Theory of Risk Bearing, Markham Publishing Co., 1971.
- [2] Burness, H. S., "On the Taxation of Non-Replenishable Natural Resources," Journal of Environmental Economics and Management, 3 (1976).
- [3] Cummings, R. G., and O. R. Burt, "The Economics of Production from Natural Resources," American Economic Review, 59 (1969).
- [4] Gordon, R. L., "A Reinterpretation of the Pure Theory of Exhaustion," Journal of Political Economy, 7 (1967).
- [5] Hendry, J. B., "The Bituminous Coal Industry," in the The Structure of American Industry, 1960, (Walter Adams, ed.), The MacMillan Co., New York (1961).
- [6] Hotelling, H., "The Economics of Exhaustible Resources," Journal of Political Economy, 39 (1931).
- [7] Kuller, R. G., and R. G. Cummings, "An Economic Model of Production and Investment for Petroleum Reservoirs," American Economic Review, 64 (1975).
- [8] Lewis, T. R., "Attitudes Towards Risk and the Optimal Exploitation of an Exhaustible Resource," Journal of Environmental Economics and Management, forthcoming.
- [9] Lewis, T. R., "Energy vs. the Environment," Social Science Working Paper No. 126, California Institute of Technology, (1976).
- [10] Sandmo, A., "On the Theory of the Competitive Firm Under Price Uncertainty," American Economic Review, 61 (1971).
- [11] Schulze, W. K., "The Optimal Use of Non-Renewable Resources: The Theory of Extraction," Journal of Environmental Economics and Management, 1 (1974).
- [12] Scott, A. T., The Theory of the Mine Under Conditions of Certainty (Mason Gaffney, ed.), University of Wisconsin Press (1967).
- [13] Smith, V. L., "Economics of Production from Natural Resources," American Economic Review, 58 (1968).
- [14] Smith, V. L., Investment and Introduction, Harvard University Press, 1961.
- [15] Symposium on the Economics of Exhaustible Resources, The Review of Economics Studies, G. M. Heal and D. F. Hendry, 1974.